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SHORT COMMUNICATION

STREAM FUNCTION SOLUTION OF NAVIER–STOKES PROBLEMS WITH INTER-ELEMENT PENALTIES

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SUMMARY

In this note, we apply a finite element stream function formulation with inter-element penalties to the Navier–Stokes equations. The approach is an extension of a technique previously introduced for Stokes. flow. The solution is obtained by iterative linearization using successive approximation, and results for a standard numerical test case are given.

KEY WORDS Finite Elements Interface Penalties Viscous Flow

DISCUSSION

The stream function form of the stationary Navier–Stokes equations in flow domain Ω is

$$-\nu\Delta^2\psi + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = f, \tag{1}$$

where ν is the viscosity and f is the applied body force. A weak variational form is easily developed using a weighted residual approach, and we have: find ψ satisfying the essential boundary conditions and such that

$$-\nu\int_\Omega^\nu \Delta\psi\,\Delta w\,dx\,dy - \int_\Omega(\psi_y w_x - \psi_x w_y)\Delta\psi\,dx\,dy = \int_\Omega f w\,dx\,dy \tag{2}$$

for all admissible test functions w with $w=0$ and $\partial w/\partial n = 0$ on that part of the boundary $\partial\Omega$ where essential conditions apply.

Let the flow domain be discretized as a union of elements Ω_e, $e = 1, 2, \ldots, E$ with $\{\chi_i\}$ the global finite element basis defining the approximation space H^h. An approximate statement of the viscous flow problem may be obtained by replacing ψ and w by ψ_h and $w_h \in H^h$ in equation (2). In the present analysis, we are particularly interested in using non-conforming elements in which the usual explicit global C^1 continuity is relaxed and H^h may not be a subspace of the solution space.[1,2] Rather than enforce the continuity of the normal derivative across the interface between adjacent elements explicitly, we add a penalty term to the variational problem to obtain

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$$\sum_{e=1}^{E}\int_{\Omega_e}\{-\nu\Delta\psi_h\Delta w_h-[(\psi_h)_y(w_h)_x-(\psi_h)_x(w_h)_y]\Delta\psi_h]\,dx\,dy$$
$$+\sum_{s=1}^{S}\frac{1}{\varepsilon}I_s([\![\partial\psi_h/\partial n]\!][\![\partial w_h/\partial n]\!])=\sum_{e=1}^{E}\int_{\Omega_e}f w_h\,dx\,dy \qquad (3)$$

for all admissible w_h. Here, $\varepsilon > 0$ is the penalty parameter and for $\varepsilon \to 0$ the added penalty term on element sides $s = 1, 2, \ldots, S$ approximately enforces the inter-element constraint condition $[\![\partial\psi_h/\partial n]\!] = 0$ on the inter-element sides Γ_s, where $[\![\cdot]\!]$ denotes the interface jump. Note that in this modified variational functional, reduced integration is used for this penalty term. That is, for given functions g, $I_s(g)$ is a low-order quadrature approximation to $\int_{\Gamma_s} g\,ds$. In a previous analysis for the linear problem,[3] it was shown that the choice of quadrature order may be based on rank conditions for an associated hybrid method and thus to the degree of the element basis chosen.

In the solution algorithm the non-linear term is treated iteratively using a successive approximation scheme; that is, in each iteration we linearize this term as

$$\int_{\Omega_e}[(\psi_h^{(n-1)})_y(w_h)_x-(\psi_h^{(n-1)})_x(w_h)_y]\Delta\psi_h^{(n)}\,dx\,dy,$$

where n is the iteration index. The remaining contributions to the linearized system at each iteration are precisely those for the Stokes flow and hence need not be re-evaluated.

NUMERICAL EXPERIMENT

As a test problem, we consider the familiar lid-driven cavity example in which the problem is defined by equation (1) in the unit square $(0, 1) \times (0, 1)$ together with the boundary conditions: $\psi = 0$ on $\partial\Omega$; $\psi_n = 0$ on the sides $x = 0$, $x = 1$, $y = 0$; $\psi_n = 1$ on the side $y = 1$. The unit square domain is discretized to a uniform mesh of right cubic (non-conforming) triangles, and the inter-element penalty integrals are calculated using a 1-point Gauss rule. The penalty parameter for this calculation was taken as $\varepsilon = 10^{-3}$. Further details on inter-element penalties and on the choice of penalty parameter, as well as its effect on the precision of the calculations, are given in References 4 and 5.

The midplane velocity profiles are plotted in Figure 1 for flow at Reynolds numbers of 100 and 400, using a uniform 6×6 mesh. As indicated in the Figure, these results compare reasonably well with those of Burggraff[6] using a finite difference solution, and with other primitive variable finite element results.[7–9] The scheme converged after 11 iterations for the case $Re = 100$ and 18 iterations for $Re = 400$ using a tolerance of $\tau = 10^{-3}$ on the relative error of successive iterates. In each case, the starting iterate is Stokes flow.

We remark that in the above calculations there is no penalty enforcement of normal derivative conditions on the exterior boundaries, but rather the data are specified as nodal point values with $\psi_x = 0$ at the corner points $(0, 1)$ and $(1, 1)$ and $\psi_y = 1$ on $y = 1$ for $0 \leqslant x \leqslant 1$.

CONCLUDING REMARK

The idea of using inter-element penalties, appropriately underintegrated, to smooth the approximate solution to the fourth-order problem appears practical and overcomes some of the difficulties associated with the use of non-conforming elements. Application of this procedure to the quadratic triangle and biquadratic quadrilateral merit investigation.

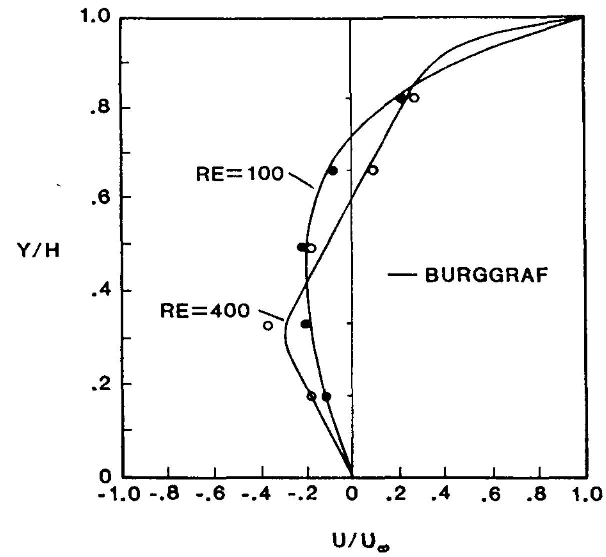

Figure 1. Midplane velocity profiles for flow in a cavity at $Re = 100$ and $Re = 400$

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